ON THE GEOMETRIC STRUCTURE OF AN ELEMENTARY CHARGE

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Iu. A. BUEVICH
(Moscow)
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The solution of the gravitational equations for an arbitrary spherically symmetric system of bodies defining the gravitational and electromagnetic fields outside this system is written out. The difficulties occasioned by the nonvariance of the energy-momentum of an elementary charge under transformations of the Lorentz group and by the divergence of the energy of the field generated by the charge are eliminated within the framework of classical field theory. It is shown that the entire mass of the charge is of field origin and that the charge itself can be interpreted as a singularity of the space-time metric.

1. The gravitational field of a spherically symmetric charged system is described by the familiar Reissner-Nordström solution. Let us write out this solution in a form conveniently suited to our subsequent discussion. Introducing the spherical space coordinates r, θ, φ we can express the square of an interval in four-dimensional space time as

$$ds^2 = e^{\nu}(dx^{\circ})^2 - e^{\lambda}dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{1.1}$$

where $x^{\circ} = ct$, c is the velocity of light, and ν , λ are some functions of t, r.

The corresponding covariant components of the metric tensor are

$$g_{11} = e^{\lambda},\ g_{22} = r^2, \quad g_{33} = r^2\sin^2\theta,\ g_{00} = -e^{\nu}$$

$$g^{11} = e^{-\lambda}, \quad g^{22} = r^{-2}, \quad g^{33} = r^{-2}\ \sin^{-2}\theta, \quad g^{00} = -e^{-\nu} \tag{1.2}$$

The tensor of the free electromagnetic field in the presence of spherical symmetry has just two nonzero components,

$$F_{10} = -F_{01} = E$$

where E is the intensity of the electrostatic field. We infer from this that the only components of the energy-momentum tensor are

$$T_0^0 = T_1^1 = -\frac{1}{8\pi}e^{-\nu-\lambda}E^2,\ T_2^2 = T_3^3 = \frac{1}{8\pi}e^{-\nu-\lambda}E^2 \tag{1.3}$$

According to the Reissner-Nordström solution the quantities ν, λ and the electrostatic field intensity E occurring in relations (1.1)–(1.3) can be written in the form

$$e^{\nu} = 1 - \frac{2\beta}{r} + \frac{\alpha}{r^2}, \quad \lambda = -\nu, \quad \beta = \frac{km}{c^2}, \quad \alpha = \frac{ke^2}{c^4} \tag{1.4}$$

$$E^2 = \frac{\alpha c^4}{kr^4}, \quad E = \pm\left(\frac{\alpha}{k}\right)^{1/2}\frac{c^2}{r^2}$$

Here m is the total mass and e the total electrical charge (which can be either positive or negative). The quantity k in (1.4) is the gravitational constant.

From (1.4) we infer that the metric is Galilean not only at infinite distances from the system of bodies and charges generating the field, but also at $r = r_0$, where

$$r_0 = \frac{\alpha}{2\beta} = \frac{e^2}{2mc^2} \tag{1.5}$$

(we must, of course, bear in mind the fact that all of the above relations hold outside the indicated system only).

If $\beta^2 - \alpha > 0$ $(km^2 > e^2)$, the metric has a singularity at $r = r_1$ and $r = r_2$, where

$$r_{1,2} = \beta \pm (\beta^2 - \alpha)^{1/2} \tag{1.6}$$

represent the "gravitational radii" of the system, and where the frame of reference with the indicated coordinates turns out to be unrealizable in the range $r_2 \leqslant r \leqslant r_1$. As we know, fixes in this range can be taken only by bodies moving in a certain way. The gravitational radii vanish for $\beta^2 - \alpha < 0$.

In the range $r > r_1$ the physical radial distance $d\rho$ corresponding to the change dr in the coordinate r and the physical time interval $d\tau$ corresponding to dt satisfy the relations

$$d\rho = \frac{r dr}{(r^2 - 2\beta r + \alpha)^{1/2}} > dr, \quad d\tau = \left(1 - \frac{2\beta}{r} + \frac{\alpha}{r^2}\right)^{1/2} dt < dt \tag{1.7}$$

The physical distance along any "circle" $r = \text{const}$ coincides with the distance computed formally from (1.1).

2. Let us use the above relations to investigate the structure of an elementary source of an electromagnetic field. We shall regard the field of such a source defined by the relations of Sect. 1 as some inclusion in an infinite pseudo-Euclidean space with a Galilean metric. Such an approach ensures that the field and the elementary source itself satisfy all the requirements imposed by the special theory of relativity. Specifically, the elementary charge considered as a material particle must of necessity be a point charge [1]. Moreover, its total energy-momentum vector must be a true four-dimensional vector invariant under linear transformations of the coordinates, and particularly under Lorentz transofmation in the unbounded pseudo-Euclidean space introduced above. In addition, the energy of the elementary charge must be finite and its momentum in the coordinate system in question must be equal to zero. These requirements were the basis of numerous attempts to construct a theory of field or nonfield mass of an elementary charge and a theory of the electromagnetic structure of elementary particles [2]. We note that the whole of the discussion to follow lies within the bounds of classical field theory, and that the possible quantum properties of the elementary charge are disregarded.

We begin by converting to Cartesian coordinates, in which the components of the metric tensor can be written in accordance with (1.2) and (1.4) in the form

$$g_{00} = -\left(1 - \frac{2\beta}{r} + \frac{\alpha}{r^2}\right), \quad g_{\alpha\beta} = \delta_{\alpha\beta} + A(r) n_\alpha n_\beta, \quad g_{0\alpha} = 0 \tag{2.1}$$

$$A(r) = \left(\frac{2\beta}{r} - \frac{\alpha}{r^2}\right)\left(1 - \frac{2\beta}{r} + \frac{\alpha}{r^2}\right)^{-1}, \quad n_\alpha = \frac{x_\alpha}{r}$$

Here and below the Greek-letter subscripts assume the values 1, 2, 3 and the Roman-letter subscripts the values 0, 1, 2, 3.

The four-dimensional energy-momentum vector for the field in the volume Ω bounded by the surface Σ can be written in the form

$$P^i = \frac{1}{c} \int_\Omega (-g)(T^{0i} + t^{0i})\, d\Omega = \oint_\Sigma \tau^{i0\alpha} d\Sigma_\alpha, \quad g = \det \| g_{ik} \| \tag{2.2}$$

Here T^{ik} is the energy-momentum tensor of matter as defined by (1.3); t^{ik} is the energy-momentum pseudotensor of the gravitational field; $d\Sigma_\alpha$ are elements of the surface Σ; and the quantities τ^{ikl} antisymmetric over the indices l and k are defined by the space-time metric

$$\tau^{ikl} = \frac{c^3}{16\pi k} \frac{\partial}{\partial x^m} [(-g)(g^{ik} g^{lm} - g^{il} g^{km})]$$

Making use of (2.1) and carrying out the appropriate calculations, we obtain the expressions

$$\tau^{00\alpha}=\frac{c^3}{8\pi k}\frac{A(r)x^\alpha}{r^2},\quad \tau^{0\alpha\beta}=0$$

$$\tau^{\beta\beta\alpha}d\Sigma_\alpha=\frac{c^3}{16\pi k}\left\{\frac{\partial}{\partial r}\left[B(r)-C(r)\frac{(x^\beta)^2}{r^2}\right]-\frac{2A(r)B(r)}{r}+\right.$$

$$\left.+4A(r)C(r)\frac{(x^\beta)^2}{r^2}\right\}n^\alpha d\Sigma_\alpha-\frac{c^3}{8\pi k}\frac{C(r)[1+A(r)]}{r}n^\beta d\Sigma_\beta \tag{2.3}$$

$$B(r)=1-\frac{2\beta}{r}+\frac{\alpha}{r^2},\quad C(r)=\left(\frac{2\beta}{r}-\frac{\alpha}{r^2}\right)\left(1-\frac{2\beta}{r}+\frac{\alpha}{r^2}\right)$$

(there is no summation over β in the third equation of (2.3)).

From (2.2) and (2.3) we obtain expressions for the energy and momentum of the field occupying the region (r,∞) of space,

$$E(r)\equiv cP^0(r)=\frac{\beta c^4}{k}-\frac{c^4}{2k}\left(2\beta-\frac{\alpha}{r}\right)\left(1-\frac{2\beta}{r}+\frac{\alpha}{r^2}\right)^{-1}=$$

$$=mc^2-\left(mc^2-\frac{e}{2r}\right)\left(1-\frac{2km}{c^2r}+\frac{ke^2}{c^4r^2}\right)^{-1} \tag{2.4}$$

$$P^\alpha(r)\equiv 0$$

in deriving which we made use of expressions (1.4) for α and β.

Hence, in accordance with the requirements of the special theory of relativity the momentum of an elementary charge in the coordinate system under consideration is in fact equal to zero.

The relativistic covariance of an elementary charge and its field, i.e. the invariance of P^i under Lorentz transformation, depends on the fulfillment of the necessary and sufficient conditions

$$\frac{1}{c}\int_\Omega(-g)(T^{\beta\beta}+t^{\beta\beta})\,d\Omega=\oint_\Sigma\tau^{\beta\beta\alpha}d\Sigma_\alpha=0 \tag{2.5}$$

for all β. This is the significance of Laue's familiar theorem [2]. By virtue of spherical symmetry we need merely require that (2.5) hold for a single β, e.g. for $\beta=1$. Converting to spherical coordinates,

$$n^\alpha d\Sigma_\alpha=r^2\sin\theta d\varphi d\theta,\quad n^1d\Sigma_1=r^2\sin^3\theta\cos^2\varphi d\varphi d\theta$$

and integrating, we obtain the following expression for the integral over the sphere $r=$ = const;

$$\int_{r=\text{const}}\tau^{11\alpha}d\Sigma_\alpha=\frac{c^3}{4k}\left\{r^2\left[\frac{dB}{dr}-\frac{1}{3}\frac{dC}{dr}\right]-2rAB+\frac{4}{3}rAC-\right.$$

$$\left.-\frac{2}{3}rC\right\}=\frac{2}{3}\frac{\beta c^3}{k}\left(-1+\frac{\beta}{r}-\frac{1}{2}\frac{\alpha}{r^2}\right) \tag{2.6}$$

From this we see that if all real space corresponds to the range $(0,\infty)$ of variation of the coordinate r, then the Laue theorem is not fulfilled explicitly. Let us suppose therefore that all space corresponds to the range $r_e\leqslant r\leqslant\infty$, so that only the values of r from this range have physical meaning; moreover, let us define r_e in such a way as to ensure fulfillment of the criteria of relativistic covariance of an elementary charge (Eqs. (2.5)). Making use of (2.6) and recalling that all space is bounded by the surface Σ consisting of an infinitely distant sphere and the sphere $r=r_e$, we obtain

$$r_e=\frac{\alpha}{2\beta}=\frac{e^2}{2mc^2}=r_0 \tag{2.7}$$

According to (2.4) the energy concentrated in all space is then diven by

$$E = mc^2 \tag{2.8}$$

This implies, among other things, that the entire mass of an elementary charge is of purely field origin.

Thus, according to the theory just developed a charge e is a singularity of the space-time metric such that the time at this singularity passes in the same way as at a distance from it, and such that the length of the elementary circle contracted to the "point" $r = r_0$ is equal to $2\pi r_0^2$ (the area of the corresponding elementary sphere is equal to $4\pi r_0^2$). The distance from this point to some other point for which the radial coordinate has some value r is readily obtainable from (1.7) and is of the form

$$\rho(r) = \int_{r_0}^{r} \frac{r\,dr}{(r^2 - 2\beta r + \alpha)^{1/2}} = \int_{r_0}^{r} \frac{r\,dr}{[r^2 + \alpha(1 - r/r_0)]^{1/2}} \tag{2.9}$$

This enables us to express the electrostatic field as

$$E = \frac{e}{\varepsilon(\rho)\,\rho^2}, \quad E\,|_{\rho=0} = \frac{4m^2c^4}{e^3} \tag{2.10}$$

Thus, the field intensity at small distances deviates from Coulomb's law; the quantity $\varepsilon(\rho)$, which can be readily determined by comparing (2.10) with (1.4) and applying (2.9), plays the role of the permittivity of vacuum ($\varepsilon(\rho) \to 1$, $\rho \to \infty$, $\varepsilon(\rho) \to \infty$, $\rho \to 0$).

We note that $r_0 > r_1$ (this is a readily demonstrable consequence of (1.5) and (1,6)), i.e. that the difficulties occasioned by the appearance of gravitational radii do not arise at all in the theory. The singularity of the metric with which we identify an elementary charge is a point singularity. This means, in accordance with the requirements of the special theory of relativity, that the "extent" of a charge can be thought of only as the extent of the field which it produces.

It is of interest to calculate separately the energy of the electromagnetic field. The mass m' produced by this field alone is given by

$$m' = \frac{1}{c} P'^0 = \frac{1}{c^2} \int_{\Omega} (-g)\, T^{00} d\Omega \tag{2.11}$$

Recalling that by virtue of (1.2), (1.3), (1.4),

$$T^{00} = g^{00}\, T_0^0 = \frac{e^2}{8\pi r^4} \left(1 - \frac{2\beta}{r} + \frac{\alpha}{r^2}\right)^{-1}$$

we obtain

$$m' = \frac{e^2}{2c^2\alpha^{1/2}} \frac{1}{(1 - x^2)^{1/2}} \left[\frac{\pi}{2} - \operatorname{arc\,tg} \frac{1 - 2x^2}{2x(1 - x^2)^{1/2}}\right] \approx$$

$$\approx m(1 + {}^2/_3 x^2) > m, \quad x = \beta\alpha^{-1/2} \tag{2.12}$$

The quantity x can be considered small (in the case of an electron $x \sim 10^{-22}$). Expression (2.12) then implies that $m' > m$, i.e. that the energy of the gravitational field produced by the electromagnetic field of an elementary charge in a sense depletes the energy of the latter. This allows us formally to assign negative mass and energy to the gravitational field generated by a charge. It is as though some part of the energy were "expended" on the appearance of the indicated singularity of the metric and on the deviation from pseudo-Euclidean character in its vicinity. We not that despite the relative weakness of the gravitational field of an elementary charge, allowance for this field is absolutely necessary to ensure fulfillment of the requirements stated at the beginning of

Sect. 2.

We have made use of the gravitational field pseudotensor in the form proposed by Fok, Landau and Lifshits; the introduction of this pseudotensor is tantamount to assuming the impossibility of localization of gravitational energy in space. There have been many studies (e. g. see [3]) in which the localization of this energy is ensured by a certain redefinition of the indicated pseudotensor. Without going into detail, we merely note that investigation of the integral momentum and integral energy is in no way hindered by the use of the above pseudotensor in preference to another, provided the coordinate system employed is Galilean at infinity (as is the case in the present study: see (2.1)).

We note that the value of r_0 in (1.5) and (2.7) coincides with the elementary charged radius given by the Bopp-Podolsky theory ($r_0 \approx 1.41 \times 10^{-13}$ cm for an electron) and agrees in order of magnitude with the effective radii obtainable in other well-known theories of charge structure (a survey of these theories appears in [2]). However, the theory developed here is quite natural in the sense that it is free of arbitrariness in the choice of the form of the electromagnetic field, Lagrangian, or, which is almost the same thing, in the choice of the form of the permittivity function (or operator). Moreover, by virtue (2.10) the form of this function turns out to be quite definite. It is not difficult to show that this function $\varepsilon(\rho)$ can be used as a basis for constructing a nonlinear electrodynamics analogous in meaning to the Born-Infeld and Schrödinger theories. This can be done simply by considering the charge in ordinary plane space, but replacing Coulomb's law by (2.10); the quantity $E|_{\rho=0}$ in (2.10) plays the role of the "maximum intensity" E_0 in the indicated theories.

We note in conclusion that for $\alpha = 0$ conditions (2.5), which follow from the Laue theorem, cannot be fulfilled for any r_e in the range $(-\infty, \infty)$. This implies that a relativistically covariant "elementary" (i.e. nonextended) material mass cannot exist. The same conclusion was arrived in a somewhat different way by Staniukovich [4], who developed the original idea of Landau.

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